Measurement of the slope of an unsteady liquid surface along a line by an anamorphic schlieren system

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Abstract. We describe and test a simple quantitative schlieren method for measuring the slope of an unsteady free liquid surface along a straight segment. The optical system, based on an anamorphic element placed at the focal plane of a standard objective, forms a distorted image of the illuminated segment on a screen. The abscissa of the resulting bright curve is proportional to the coordinate along the segment and the ordinate to the longitudinal slope at the corresponding coordinate, provided that the transverse slope component is negligible. The method is useful to probe axisymmetric flows along a diameter or flows in straight channels along the axis.

1. Introduction

When studying the spreading of drops, it is often important to determine the time-dependent slope of the free liquid surface of a meridian section [1–5]. Here we describe a simple, cheap and practical quantitative schlieren method giving without scanning the slope in a particular direction, say \( \partial h/\partial x \) as a function of the coordinate \( x \) in the same direction. The method may also be useful to probe other unsteady liquid surfaces, such as those in channel flows.

Schlieren methods [6, 7] are powerful tools for detecting small ray deviations, which are in principle adaptable to our problem. A parallel probing beam crosses (say in the \( z \) direction) both a high-quality optical substrate (parallel to the \( x-y \) plane) and a supported liquid layer of variable thickness \( h(x, y; t) \), so that all the ray deviations occur only at the liquid surface. A lens \( L \) conjugates the liquid surface with a screen where these deviations are converted into measurable magnitudes thanks to some spatial filtering at the focal plane \( \Phi \) of \( L \). Simple techniques based on the Toppler knife-edge method [6, 8] need, however, accurate calibrations to extract quantitative information from the intensity distribution. On the other hand, the defocused grid technique [6] gives slopes \( \partial h/\partial x \) only at a discrete set of points along \( x \).

An attractive approach is to obtain on a screen the Cartesian representation of \( (\partial h/\partial x) \) versus \( x \) without scanning. This representation may be provided, for instance, by the inclined slit technique [6,9,10]. The refracting object is illuminated by a beam with large aperture in the \( y-z \) plane and that is almost parallel in the \( x-z \) plane. Then an inclined slit is placed at the focal plane \( \Phi \) of \( L \) and a cylindrical lens with its axis in the \( x \) direction conjugates the slit with the screen. A bright curve \( y_s \) versus \( x_s \) appears there, with \( x_s \) and \( y_s \) proportional to \( x \) and \( \partial h/\partial x \), respectively. A similar curve may be produced by the crossed prisms method [6], based on the selection of colours from a white light beam. Different versions [6,11–14] both of the inclined slit and of the crossed prisms method have been proposed; the common feature is that the object must be illuminated with a beam of aperture larger than the largest deviation to be measured. As discussed later, this may be a drawback for the application of our interest, in which the deviations are often rather large.

In the present work we show that the above Cartesian representation may be obtained in a quite different way, which, as far as we know, has not previously been described and is advantageous in many respects. The liquid is illuminated by a parallel light sheet along a narrow straight segment and, by means of an anamorphic system, a distorted image of the segment is formed on the screen. This anamorphic optical system is designed in such a way that \( y_t \) is proportional to the slope \( \partial h/\partial x \) rather than to the coordinate \( y \), as happens with a standard lens. Although, in the above methods, only a small fraction of the light leaving the fluid forms the curve on the screen (namely
the light which passes through the inclined slit), in the present method almost all this light is concentrated there. In section 2 we show that a suitable anamorphic system may be set up simply by placing at the focal plane of L an element A equivalent to a doublet formed by a positive and a negative crossed cylindrical lens. This system may be considered as a rigid mechanical unit, since the relative positions of L and A are fixed. As described in section 3, the method has been successfully tested by measuring the slope of circular droplets along their diameters. For the case of small drops, the results are directly compared with data from interferometric measurements. For larger drops, for which interferometry is unwieldy, liquid masses corresponding to the volumes calculated from the slope measurements are compared with the masses given by a precision balance. Finally, the main qualities of the method are summarized in section 4.

2. The optical system

2.1. The basis of the method

We consider first an ideal parallel light sheet of finite width and negligible thickness. The z axis is taken as coincident with the central ray of the sheet (see figure 1). At z = 0, ‘the object plane’, there is an optically thin phase object. The x axis in this plane is taken along the sheet width. A lens L aligned with the z axis conjugates the object plane with a screen (image plane). We define the coordinate systems x, y on the image plane and x_f, y_f on the focal plane \( \Phi \) parallel to x and y, respectively. The ray deviations produced by the phase object are assumed to be contained within the x–z plane with angles with respect to the x axis given by \( \alpha_x(x) \), namely the transverse deviation angles \( \alpha_x \) are assumed to be zero. Therefore, the rays cross the \( \Phi \) plane along \( x_f \) at positions depending on \( \alpha_z \). In this plane we place an anamorphic phase-shifting element A whose properties on the axis \( x_f \), are that (i) the optical thickness \( t \) is uniform (\( \partial t/\partial x_f = 0 \) for \( y_f = 0 \)), (ii) \( \partial t/\partial y_f \) is proportional to \( x_f \), and (iii) \( \partial^2 t/\partial y_f^2 = 0 \) (this last condition is convenient but not strictly necessary). Clearly, such an anamorphic element does not change the relationship between the coordinates \( x_s \) on the screen and \( x \) on the object. However, due to the dependence of \( \partial t/\partial y_f \) on \( x_f \), the rays are deviated from the x–z plane by angles proportional to their coordinate \( x_f \). Therefore, a bright curved segment appears on the screen displaying the function \( \alpha_z(x) \) with scale factors depending on the properties of the optical system.

The required conditions are produced by an anamorphic element equivalent to a doublet formed by a positive and a negative cylindrical crossed lens with their axes along the Cartesian coordinates \( \xi \) and \( \zeta \). In fact, the optical thickness \( t \) of this doublet is

\[
t(\xi, \zeta) = t_0 + \frac{1}{2} f_1 \xi^2 - \frac{1}{2} f_2 \zeta^2
\]

where \( f_1 \) and \( f_2 \) are the absolute value of the focal distances corresponding to each one of the doublet components. If

the axes \( f \) and \( f \) axes of the \( \phi \) plane, respectively, the optical thickness may be expressed in terms of \( x_f \) and \( y_f \) as

\[
t(x_f, y_f) = t_0 + \frac{1}{2} \left( \cos^2 \theta - \sin^2 \theta \right) x_f^2
\]

\[
+ \frac{1}{2} \left( \sin^2 \theta - \cos^2 \theta \right) y_f^2
\]

\[
+ \left( \frac{1}{f_1} + \frac{1}{f_2} \right) x_f y_f \sin \theta \cos \theta.
\]

For the particular angle \( \theta = \theta_p \) defined by

\[
f \cos^2 \theta_p = f_1 \sin^2 \theta_p
\]

equation (2) reduces to

\[
t(x_f, y_f) = t_0 + \frac{1}{(f_1 f_2)^{1/2}} x_f y_f + \frac{f_2 - f_1}{2 f_1 f_2} y_f^2.
\]

Therefore, the optical thickness is given by a symmetric hyperbolic term and a quadratic term equivalent to the thickness of a cylindrical lens with its axis along \( x_f \). This last term vanishes for \( |f_1| = |f_2| \) (\( \theta_p = \pi/4 \)) and then the symmetry with respect to the two axes is complete.

For this simplest case, the conditions (i)–(iii) are satisfied and the element may be thought of as formed by a linear array of triangular prisms disposed with their edges parallel to the \( x_f \) axis and wedge angles depending linearly on their position \( x_f \), arranged in such a way that the optical thickness is uniform along the axis \( x_f \).

Now, by considering the complete optical system, a light ray leaving the object plane at coordinates \((x, y)\) at angles \((\alpha_x, \alpha_z)\) with respect to the z axis impinges upon the screen at

\[
x_s = M \left( -x + \frac{F^2}{f} \alpha_y \right)
\]

\[
y_s = M \left( -y + \frac{F^2}{f} \alpha_x \right)
\]

where \( F \) is the focal length of the lens L, \( f = (f_1 f_2)^{1/2} \) (in the particular case that we are considering \(|f_1| = |f_2| = f \)) and \( M \) is the magnification of the system generated by L alone. Therefore, when a plane light sheet containing the x axis \( (y = 0) \) crosses an optically thin phase object which deviates the rays only in the x direction \( (\alpha_x = 0) \), a focused distorted image \( y_s(x_s) \) of the illuminated straight line appears on the screen, with

\[
x_s = -M x
\]

\[
y_s = -M \frac{F^2}{f} \alpha_x
\]

The curve \( y_s \) versus \( x_s \) is then a Cartesian representation of the function \( \alpha_x(x) \) with proper scale factors. Note that, for a given \( F \alpha_x \), the ordinate on the screen depends both on the ratio \( F/f \) and on the magnification \( M = x_s/x \) of the optical system. Besides, the aspect ratio \( y_s/x_s \) of the representation does not depend on \( M \) but only on the ratio of the focal lengths. In consequence, by moving the set L–A and the screen along the z axis, \( M \) may be changed without modifying the aspect ratio.
2.2. Departures from the ideal system

In practice, we had to resort to the illumination system sketched in figure 1. Therefore, the beam profile on the y–z plane is basically given by the caustic curve determined by the cylindrical lens placed after the beam expander. The object is located at the focal plane of this lens, where the beam thickness has a minimum (non-zero) value \( y \) and the angular aperture \( \alpha_m \) in the y–z plane cannot be neglected. Instead, the beam profile along the x–z plane is practically parallel. Summarizing, the illuminated region is a narrow strip of width \( \Delta y \) centred on the x axis and the rays forming angles \( \alpha_y \leq \alpha_m \neq 0 \) with respect to the x–z plane. Therefore, from equation (6) the thickness of the curved line on the screen is \( \Delta x_s = M \Delta y \) and is independent of \( \alpha_m \). However, due to the finite aperture \( \alpha_m \) of the light sheet, the rays leaving the object at a given x produce on the screen not a point but rather a segment parallel to the x axis and centred at \( x_s \), whose length is

\[
\Delta x_s = M \frac{F^2}{f} \alpha_m.
\]  

(9)

In conclusion, each point in the object plane is spread into a region (\( \Delta x_s, \Delta y_s \)). Because the illuminating system may be designed close to the diffraction limit, a reduction of \( \Delta y \) requires an increase in \( \alpha_m \) and vice versa. Then, the pragmatic condition \( \Delta x_s \approx \Delta y_s \) determines the values of \( \Delta y \) and \( \alpha_m \) depending on \( F, f \) and on the light wavelength, while the magnification \( M \) gives the actual magnitudes both of \( \Delta x_s \) and of \( \Delta y_s \).

Now, let us consider the effects of eventual transverse ray deviations \( \alpha_y \). According to equation (5) \( \alpha_y \) affects the coordinate \( x \) but not \( y \). A uniform transverse deviation along the probed region merely shifts the bright curve in the screen along the x direction without distortion. However, non-uniform transverse deviations \( \alpha_y(x) \) modify the linear correspondence between \( x_s \) and \( x \). This limits the application of the method to situations with well defined geometries (a limitation which also exists for the other methods quoted in section 1). For instance, probing should be performed along the diameter of a drop or along the axis of a channel. It is noteworthy that the presence of transverse deviations could be noted by observing the light distribution at the plane \( \Phi \) and that the correspondence between \( x_s \) and \( x \) could in principle be restored from this information. In fact, from \( y_s \) we get \( x_s \) without distortion and the distribution at the focal plane gives \( y_s \) as a function of \( x_s \), which, on turn, determines the value of \( x_s \) corresponding to \( x \) through equation (5).

Even though the use of an anamorphic element with \( |f_1| = |f_2| \) is preferable, the conditions (i) and (ii) of section 2.1 are still fulfilled for \( |f_1| \neq |f_2| \), provided that the element is rotated by an angle \( \theta_p \neq \pi/4 \) (see equation (3)). In this case, \( x_s \) is still given by equation (5) and (6) becomes

\[
y_s = M \left( -y + \frac{F^2}{f} \alpha_x - \frac{f_2 - f_1}{f^2} \alpha_y \right).
\]  

(10)

For the ideal case \( y = 0, \alpha_y = 0 \) both coordinates \( x_s \) and \( y_s \) on the screen are still given by equations (7) and (8). However, some differences appear for a light sheet with \( y \leq \Delta y \neq 0 \) and \( \alpha_y \leq \alpha_m \neq 0 \). It is easy to see that the image plane defined by L alone does not coincide with the image plane defined by the set L–A in the y direction and...
hence the curved segment on the screen appears defocused. According to equation (5), the size $\Delta x_0$ is the same as before (equation (9)), but $\Delta y_0$ depends both on $\Delta y$ and on $\alpha_m$. In consequence, an anamorphic element with equal focal lengths produces on the screen a bright line thinner than a non-compensated element. The image for the case $|f_1| \neq |f_2|$ can still be focused by adding a corrective cylindrical lens. However, because the determination of the central coordinates of the curve is not affected very much by the curve thickness, the correction may be omitted even for a strongly non-compensated anamorphic element.

3. Testing the method

3.1. The actual optical system

Light from a He–Ne laser is expanded and then a cylindrical lens with large focal length forms a light sheet with aperture $\alpha_m \approx 10^{-2}$ rad, negligible lateral aperture, width about 5 cm and minimum thickness $\Delta y \approx 10^{-2}$ cm at the object plane (figure 1). The beam is sent orthogonally through a plane-parallel glass substrate which supports the liquid to be probed, namely droplets of silicone oil (refractive index $\eta \approx 1.4$). A micrometer device allows alignment of the probing light sheet with a drop diameter, so that $\partial h/\partial x = 0$ can be assumed along the probed region. The direction $\alpha_x$ of a ray leaving the liquid at $x_0$ is simply related to the liquid slope $\partial h/\partial x$ by Snell’s law, which is used in the exact form in our calculations, even though the approximate equation $\alpha_x \approx (\eta - 1)\partial h/\partial x$ is accurate enough for $\alpha_x < 0.1$. The thickness differences within the probed region are so small ($h \approx 10^{-1}$ cm) that the free surface may be considered to be at $z = 0$, namely at the object plane. This plane is conjugated with a screen by the sets L–A, whose optical parameters are given below.

To make validation of the results possible, we prepared an arrangement allowing easy conversion from a Mach–Zehnder interferometer [7] to the schlieren system. The conversion must be done in a short time (a few seconds in our case), because we need almost simultaneous records of the interferograms and of the slope curves. Besides, interferometry requires thin drops, so the drops were left to spread for several hours before the measurements. The horizontality of the substrate was accurately checked in order to preserve the circular symmetry of the spreading. The records were made with a Sony CCD camera, digitized by using a frame-grabber (Peplus provided by Imaging Technology Inc, 512 x 512 pixels) and processed by using standard software.

For the present work we use two anamorphic elements A. The first is an ophthalmic lens with spherical correction $+1.5$ dioptres and cylindrical correction $-3$ dioptres, with measured focal lengths $f_1 = 65$ cm, $f_2 = -71$ cm. The second is a lens with crossed cylindrical surfaces, with measured focal lengths $f_1 = 17$ cm and $f_2 = -9.5$ cm. Therefore, only for the first anamorphic element do we have $|f_1| \approx |f_2|$. We used alternatively two high-quality objectives with focal lengths $F = 17.8$ and 40 cm for the lens L. Therefore, we tested four different sets L–A covering a considerable range of optical parameters, in particular the ratio $F/f$ could be varied in the range 0.26–3.15. Within the used optical magnification range ($3 \leq M \leq 8$), slopes of the liquid surface of $10^{-3}$ rad produced observable shifts of several millimetres on the screen.

We calibrated the slope scales by probing an inclined large cell (to reduce capillary effects) containing the same liquid that had been used for the drops. A micrometer screw allows determination of the inclination angle $\omega$ to within $10^{-4}$ rad. The positions of the line on the screen (in this case straight lines parallel to the $x_0$ axis) for several values of $\omega$ are shown in figure 2. Note the linearity of the relationship $y_s(\omega)$ and also the large range of sensitivities covered.

3.2. Results

Figure 3 corresponds to a droplet spreading on a horizontal glass substrate (sessile drop) [1–3, 15]. The small circles give the slope $h' = \partial h/\partial x$ obtained from the recorded curves (the coordinates correspond to the central points of the bright curve on the screen) and the small squares correspond to the liquid height $h$ calculated by numerical integration. The large squares refer to the liquid height $h$ measured from the interferograms (each point corresponds to a fringe) and the large circles to the slopes calculated by numerical derivation. For a sessile drop the slope of the free surface increases monotonically from the centre to very near the contact line, where it jumps quite abruptly to the zero slope corresponding to the uncovered substrate. From a practical point of view, the jump helps one to ensure proper alignment of the anamorphic element. In fact, an incorrect rotation ($\theta \neq \theta_p$) leads to a difference between the abscissae on the two sides of the jump.

Figure 4 shows the experimental data for a droplet hanging from a horizontal glass substrate (a hanging drop).
Figure 3. Results for a sessile drop of volume $4.97 \text{ mm}^3$. The large squares give heights obtained from interferometry and the large circles the corresponding slopes calculated by numerical derivation. The small circles are the slopes given by the schlieren method and the small squares the corresponding heights calculated by numerical integration ($F = 17.8 \text{ cm}$, $f = 12.7 \text{ cm}$ and $M = 6.91$).

Figure 4. Results for a small hanging drop of volume $0.465 \text{ mm}^3$. Like in figure 3 the large symbols correspond to interferometry and the small symbols to the schlieren method ($F = 40 \text{ cm}$, $f = 12.7 \text{ cm}$ and $M = 3.64$).

Figure 5. Results for a hanging drop of volume $3.91 \text{ mm}^3$. The large symbols are the data from interferometry and the lines are from the schlieren method. The different curves result when the incident light sheet is shifted by 0.005, 0.005 and 0.015 cm from the drop diameter ($F = 17.8 \text{ cm}$, $f = 12.7 \text{ cm}$ and $M = 6.93$).

Table 1. Masses of four hanging drops expressed in grams. Weighed masses were obtained with a Metler balance model H35AR with an error of $\pm 0.0002 \text{ g}$. Values of the mass calculated from the profile were determined by the schlieren method (liquid density $0.967 \text{ g cm}^{-3}$, $F = 17.8 \text{ cm}$, $f = 67.9 \text{ cm}$ and $M = 7.01$).

<table>
<thead>
<tr>
<th>Drop</th>
<th>Weighed mass (g)</th>
<th>Calculated mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0214</td>
<td>0.02216</td>
</tr>
<tr>
<td>2</td>
<td>0.0162</td>
<td>0.01620</td>
</tr>
<tr>
<td>3</td>
<td>0.0206</td>
<td>0.02015</td>
</tr>
<tr>
<td>4</td>
<td>0.0156</td>
<td>0.01534</td>
</tr>
</tbody>
</table>

In figure 5 we show how a shift between the light sheet and a drop diameter results in non-symmetric curves on the screen due to the distortion of the $x$ scale caused by a non-uniform transverse slope. A practical way to obtain a good alignment is provided by the observation of the light pattern in the plane $\Phi$: the bright spot due to rays passing outside the drop must fall just on a straight bright segment formed by the rays refracted at the liquid surface.

For larger drops [15] validation by interferometry is unwieldy because of the large number of fringes. Therefore, in table 1 we compare drop masses calculated from the height profile with masses measured by a precision balance. The differences are less than 1%, which is within the range of the instrument’s precision.

4. Final remarks

The system described here may be suitable for many applications due to the following features.

(i) For the same total incident intensity and magnification, the curve on the screen is much brighter than those obtained by previous methods. For this reason the system is good for probing surfaces by reflection. This may be of interest not only when the optical properties either of the liquid or of the substrate make the refraction mode of operation impossible, but also because of the increase in the
sensitivity by an approximate factor of $2/(\eta - 1) \approx 5$ in our case.

(ii) The incident beam is almost parallel and hence the incidence may be made virtually normal for all the rays. Previous methods require incident beams of rather large aperture when probing free liquid surfaces, thus leading to a change in the size of the probed surface region with the thickness $h$. Moreover, in these methods the curve on the screen is formed by rays whose angle of incidence is a function of the ordinate. As a consequence, the contribution to the deviation due to surfaces other than the free liquid surface may be appreciable and different for each value of $y$.

In this work we measured slopes up to values as high as $\partial h/\partial x \approx 0.25$ ($\eta = 1.4$). This range may be extended by using an objective $L$ with higher numerical aperture and a larger anamorphic element A. In conclusion, the convenient features of the method, namely the precision of the results and the wide range of measurable slopes, make this schlieren system an interesting tool for measuring the slope of unsteady free liquid surfaces. As shown, the method may also be used to obtain accurate values of the liquid thickness by numerical integration.

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References

[10] Philpot J St L 1938 Direct photography of ultracentrifuge sedimentation curves Nature 141 283–4