Instabilities in the flow of thin films

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We analyze theoretically and computationally the contact line instabilities that develop as a fixed volume of liquid flows down an incline. Analogously to the flow of a continuous stream of liquid (constant flux case) studied previously (Phys. Fluids 13, 3168 (2001)), we perform the study within the lubrication approximation (thin film flow). We find that the fact that only finite amount of liquid is available significantly modifies some aspects of the instability development, although the main features of these two flows are rather similar. The similarities and the differences are analyzed and explained using analytical (linear stability analysis) and computational (fully nonlinear time-dependent simulations) methods.

Thin film flow is relevant in many situations, including various coating flows, cooling systems in nuclear reactors, or lung airways. These flows can be driven by gravity (flow down an inclined plane), Marangoni, or chemical forces. Balance between the various forces driving the flow often leads to instabilities which lead to corrugated contact line and formation of patterns of finger-like or triangular shape. In a number of applications, these instabilities are not desirable, since they may lead to dry regions, or to uneven substrate coverage. Therefore, it is of significant importance to understand the nature and the source of these instabilities.

In our previous works [1-3], we have addressed the instability development in the flow of a constant stream of an incompressible thin film down an incline. In that flow configuration, a constant flux (CF) of the liquid is assumed to be supplied far behind the contact line. This case has also been analyzed in some detail by other researchers, using both analytical and computational methods (see e.g., [4-9]). In particular, it has been conjectured that the instability is related to the formation of a capillary ridge in the fluid profile, just behind the advancing contact line [4,5]. This is a basic result of the linear stability analysis (LSA), which assumes that the liquid thickness is kept constant at infinity (far behind from the contact line). Most of the experiments, however, have been performed by releasing a constant volume (CV) of fluid at the top of an incline [10-12]. It is commonly assumed that by proper choice of scales, the results of the CF analysis and computations can be applied to CV experiments. The analysis of the correctness of this assumption is the main subject of this work.

Before delving into the details of the underlying mathematical equations, it is worth reviewing the main assumptions of the model used in this work. First of all, any theoretical and/or computational analysis requires resolving the so-called “contact line paradox”. As it is well known, assuming standard no-slip boundary condition at the contact line leads to a multivalued velocity field there (see e.g., [13-15]). This problem is typically approached by either relaxing the no-slip boundary condition, or by assuming a thin precursor film in front of the propagating contact line. The latter approach is chosen here, mainly due to the computational reasons outlined in some detail in an earlier work [16]. Next, the dynamics of the main body of the fluid film is described within the framework of the lubrication approximation, therefore assuming that the fluid is thin, and that all gradients are small. Finally, only completely wetting fluids are considered in this work. The reader is referred to [8,9] for simulations and analysis of partially-wetting flows.

I. BASIC EQUATIONS AND NUMERICAL METHODS

Within the framework of the lubrication approximation, the flow is described by the following governing equation for the fluid thickness $h(x,y,t)$ (see e.g. [1] and references therein):

$$
\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0,
$$

$$
\mathbf{u} = -\frac{1}{3\mu} \nabla \cdot \left[ \gamma h^2 \nabla^2 h - \rho gh^2 \nabla \cos \alpha + \rho g h^2 \sin \alpha \mathbf{i} \right],
$$

where $\mathbf{u} = (u,v)$ is the depth averaged fluid velocity, $\nabla = (\partial_x, \partial_y)$, $\mu$ is the viscosity, $\rho$ is the density, $g$ is the gravity, and $\alpha$ is the inclination angle of the plane of the substrate. The coordinate frame is chosen so that $i$ points down the incline, and $j$ is the transverse direction in the plane. As mentioned above, Eq. (1) assumes no-slip boundary condition at the fluid/solid interface. In the CV flow, this equation has to be solved under the constraint that the fluid volume

$$
V = \int_0^L \int_0^X h(x,y,t) \, dz dy,
$$

\begin{align*}
\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) &= 0, \\
\mathbf{u} &= -\frac{1}{3\mu} \nabla \cdot \left[ \gamma h^2 \nabla^2 h - \rho gh^2 \nabla \cos \alpha + \rho g h^2 \sin \alpha \mathbf{i} \right],
\end{align*}

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II. ONE DIMENSIONAL SOLUTION

If we ignore the y-dependence of the flow, we are left with a one dimensional (1D) problem. For late times, when the transient effects associated with the initial release of the fluid have decreased enough, a self-similar solution, that balances the weight of the fluid and the viscous force on the plane (i.e., the first and last terms in Eq. (5)), can be obtained. Thus, by neglecting surface tension and normal gravity effects, we obtain this solution (first derived by Huppert [10]) in the form

$$ h(x, t) = \left(\frac{x}{3t}\right)^{1/2}. $$

(8)

The domain of the solution ends abruptly at:

$$ x_H(t) = \left(\frac{27}{4} A^2 t\right)^{1/3}, $$

(9)

where the thickness is

$$ h(x_H) = h_H = (A/2t)^{1/3}, $$

(10)

and A is the dimensionless volume of fluid per unit channel width. By replacing this solution in the neglected terms, we find that it holds for (see [17]),

$$ 1 << (48x^5 t)^{1/2}, \quad D << (3xt)^{1/2}. $$

(11)

Thus, Huppert’s self-similar solution is reached asymptotically as $t \to \infty$; the smaller $\alpha$ (larger $D$), the longer it takes to reach this solution. However, as noted by Hocking [17], the main criticisms that can be made to this solution regards the validity of the lubrication approximation. It violates the small slope hypothesis both at the leading edge, where $h$ goes abruptly from $h_H$ to (practically) zero, and at the rear wall, where the slope is infinite. To proceed, one needs to include surface tension, and the normal gravity term for $\alpha < \pi/2$. This is done by adding the corresponding fourth and second order terms, as in Eq. (5). The inclusion of any of these terms breaks self-similarity.

Figure 1 shows the time evolution of the fluid profile as given by our numerical solution of the 1D version of Eq. (5) for the spreading of a given amount of fluid down a vertical plane (for brevity, we concentrate here on the case $D = 0$, i.e. $\alpha = \pi/2$). The initial condition is a step-like function smoothed by a transition zone of width $\Delta$; thus, $h(x, 0) = 1$ for $0 \leq x \leq x_0 - \Delta/2$, and

$$ h(x, 0) = \left[1 - \left(\frac{x - x_0 + \Delta/2}{\Delta}\right)^2\right]^2 + b $$

(12)

for $x_0 - \Delta/2 \leq x \leq x_0 + \Delta/2$, where $b$ is the (dimensionless) precursor film thickness. Thus, the area is $A = x_0 + \Delta/30$ where $\Delta \ll x_0$ (typically, we use
\( \Delta = 1 \). For early times, the flow develops a large bump, often called capillary ridge, in the front region, which shortly after monotonically decreases as the fluid spreads. Furthermore, the fluid thins far behind the contact line. This is in contrast to the CF configuration, in which both the size of the capillary ridge and the thickness of the fluid far behind the contact line are constant. In that configuration, a traveling wave forms, i.e., the fluid is simply translating down an incline, and the flow is translationally invariant. There is no such an invariance in the CV case.

![Graph](image)

**FIG. 1.** Thickness profile \( h(x, t) \) for \( x_f = 10 \) \((A = 10.03 \text{ with } \Delta = 1)\) flowing down a vertical plane \((D = 0)\), using a precursor film thickness \( b = 10^{-2} \) and a grid size \( \Delta x = 0.05 \).

Figure 2a shows the thickness of the bump, \( h_b \) (thick line), as a function of time; for large times, it evolves according to a power law with an exponent very close to \( 1/3 \), in agreement with the maximum thickness \( h_H(t) \) in Huppert’s solution (see Eq. (10)). However, Huppert’s solution (dashed line in Fig. 2a) is a better approximation to the asymptotic behavior of the thickness at the almost flat region just behind the bump, \( h_p(t) \) (thin line). This flat region is the transition zone between the (‘outer’) Huppert’s solution and the (‘inner’) region with the bump, where surface tension forces are dominant.

The front position \( x_f(t) \) shown in Fig. 2b is defined as the coordinate of the small depression (dip) developed ahead of the bump region [1]. For large \( t \), \( x_f \) compares very well with Huppert’s solution, \( x_H(t) \), though an exponent in the time dependence somewhat larger than \( 1/3 \) is suggested by the calculation. This indicates that the spreading rate is basically determined by the component of gravity parallel to the plane, and that surface tension has only a very small influence on it.

It has been suggested in the literature [4,18] that the traveling wave solution of the CF case represents the ‘inner’ solution of the CV configuration. To analyze this concept, we show in Fig. 3 a comparison between some of the profiles shown in Fig. 1 and the corresponding traveling wave profiles of the CF case, as a function of \( \zeta = x - x_f \). The profiles for the CF case have been appropriately scaled so that they correspond to a constant thickness \( h_p \) far behind the contact line, and whose precursor thickness is equal to \( b = 10^{-2} \) far ahead. While the agreement of both solutions in the bump region improves as time increases, for early times there is a significant difference, in particular in the size of the capillary ridge. These results suggest that the CF solution (traveling wave) is not appropriate to describe the CV flow for early times. Moreover, as shown in Fig. 1, for the early times a very large bump develops in the CV case, indicating that the flow is in fact very unstable with respect to transverse perturbations (this is also true for \( D > 0 \)). Thus, if the instability develops for early times, it does not seem appropriate to employ the CF solution as the base state to describe the onset of the instability. We shall return to this issue in the following section.

![Graph](image)

**FIG. 2.** (a) Thickness of the bump \( (h_b, \text{ thick line}) \) and of the transition region (almost flat) behind the bump \( (h_p, \text{ thin line}) \) of the profiles in Fig. 1. The asymptotic behavior of \( h_p \) is shown by the dashed line. (b) Numerical front position \( x_f \) (thick line) and \( x_H \) (dashed line), as given by Eq. (9).

Figure 4 shows the effects of the plane inclination, \( \alpha \), on the fluid profile by increasing parameter \( D \) from 0 to 5. One effect is related to the slowing down of the spreading; for a given (non-dimensional) time, \( x_f \) is smaller, as shown in Figure 4a. This smaller \( x_f \) leads to larger thickness of the bulk due to mass conservation (note the arrows in Fig. 4a). Another effect is the decreasing of the bump height and the flattening of the transition region, as shown in Fig. 4b. Note that the width of the bump region remains practically constant, and also that the departure from Huppert’s solution increases for larger \( D \)’s.
FIG. 3. Comparison between the thickness profiles of the CV case (see Fig. 1) and the corresponding traveling wave solutions of the CF case, at three different times. Part b) shows the profiles as a function of distance from \( x_f \).

FIG. 4. Thickness profiles at \( t = 100 \) for several \( D \)'s (arrows indicate increasing \( D \)). The rest of the parameters are as in Fig. 1. The dot-dashed line corresponds to Huppert's parabola.

The precursor film thickness also has an effect on the dynamics. In Fig. 5 we show, as a typical example, the fluid profile at \( t = 100 \) for \( D = 5, x_{f_0} = 5 (A = 5.03) \) and different values of \( b \). Due to the increase of viscous dissipation for smaller \( b \)'s, the fluid front propagates slower as \( b \) decreases. Consistently, the amount of fluid in the front region grows in order to balance the increase of viscous forces, thus leading to a higher bump.

FIG. 5. Thickness profiles at \( t = 100 \) for different \( b \)'s. We use \( D = 5 \) and \( x_{f_0} = 5 \). The dot-dashed line corresponds to Huppert's parabola.

III. TWO-DIMENSIONAL SIMULATIONS

In this section, we allow for the flow to develop the \( y \)-dependence. First, we discuss the LSA for the CV flow, and then present the results of fully nonlinear simulations.

A. Linear stability analysis

The LSA for the CV case can be thought of as an extension of the LSA for CF problem. In the analysis of the latter case, \( h, x \) and \( t \) are scaled by:

\[
h^{CF} = h_0, \quad x^{CF} = x_c \left( \frac{h_0}{a} \right)^{1/3}, \quad t^{CF} = t_c \left( \frac{a}{h_0} \right)^{5/3},
\]

(13)

where \( h_0 \) is the constant (dimensional) thickness far behind the front. In order to extend this scaling to the CV case, it seems appropriate to take \( h_c = h_H(t) \approx h_p(t) \) (instead of Eq. (4)) so that \( x_c(t) \) as well as \( t_c(t) \) are now time dependent*.

At least during the time interval in which the outer solution (close to Huppert's solution) does not change much, the results of the LSA for the CF case should remain valid. Now, assume that at \( t = 0 \) there is a single transverse mode present, of (dimensional) wavelength \( \lambda_{dim} \). At \( t = t_1 > 0 \) the fluid thins (\( h_H(t) \) diminishes), so that the appropriate length scale \( x_c(t_1) \)

* This choice is convenient only for this analysis, since for the purpose of comparing with experiments, it is more convenient to work in terms of fixed scales.
B. Computational results

Next, we present the results of fully nonlinear 2D simulations. These simulations are performed using as initial condition the results of 1D simulations, perturbed at \( t = 0 \) by a single transverse mode characterized by the wavelength \( \lambda_0 = L_y \). The position of the front at \( t = 0 \) is then given by

\[
x_f(y) = x_{f0} - A_0 \cos(2\pi y/\lambda_0),
\]

where \( x_{f0} \) is the unperturbed position [1,2]. The perturbation is characterized by a small amplitude \( A_0 = 0.1 \), and a phase such that this initial condition satisfies \( \partial h/\partial y = 0 \) at \( y = 0, L_y \).

Figure 6 shows the results for two representative cases, \( D = 0 \) and \( D = 1 \). From the snapshots of the contact line position of the flow down a vertical plane (Fig. 6a), we see that the main difference compared to the CF case (Figs. 4 in [1]) is the slowing down of the flow due to thinning of the fluid as time progresses. Figure 6b shows that for the flow down an inclined plane (\( D > 0 \)), the instability is weaker, again similarly to the CF case. The slowing down of the fluid is also noticeable in Fig. 6c, which shows the positions of tips and roots. When plotted on a log-log scale, these results show that \( x_t \) and \( x_r \) are well fitted by power laws \( x_t \sim t^{0.4} \) and \( x_r \sim t^{0.3} \), in agreement with the experimental data [11,12,19]. We emphasize that, even though the speed of the roots becomes rather small for later times, the roots never stop moving, similarly to the CF case [1,2]. Therefore, complete substrate coverage is obtained also in the CV case.

Figure 6d shows the length of the patterns for \( D = 0 \) and \( D = 1 \). We see approximately exponential growth for early times, and the slower growth for later times. The initial growth rate is very similar to the one calculated in the CF case, as it should be, since for these early times the fluid has not had enough time to thin significantly. For \( D > 0 \), growth rate is decreased, consistently with LSA and experiments [11]. The transition from exponential to linear growth occurs about time \( t = 20 \) for \( D = 0 \), and \( t = 50 \) for \( D = 1 \), again consistent with the experimental results [11].

We note that, due to the thinning of the fluid, the thickness of the precursor increases relative to the thickness of the body of the fluid, thus influencing the stability in the similar manner as shown in [1] for the CF case (i.e., widening the rivulets).

![FIG. 6. Snapshots of the fluid profile for CV flow down a vertical plane (a) and inclined plane (b). (c) The positions of tips and roots for \( D = 0, 1 \). (d) The lengths of the patterns for \( D = 0, 1 \).](image)

Figure 7 shows the lengths of the patterns for long times and for different \( D \)'s. Clearly, the growth slows down for larger \( D \)'s, as expected. In CF case [1], we found that growth saturation occurred, i.e., after some time period, which depended on \( D \), the growth stopped and the fluid simply propagated down the incline. In CV flow shown in Fig. 7, however, this saturation does not occur, although the growth is rather slow for long times and large \( D \)'s. For a qualitative explanation of this effect, we have to go back to the issues related to the thinning of the fluid. In terms of our “frozen” scales, the (dimensionless) length of the pattern always increases, so we do not expect saturation even for large \( D \)'s. These results are in agreement with the experimental observations [11].

To explore the influence of the area \( A \) on the development of the instability, we have performed simulations with different values of \( x_{f0} \). Just minor changes in the length of the fingers have been obtained, with stronger growth for larger volumes. Also, we have directly verified that fluid thinning can lead to instability, as predicted by LSA. The fluid is initially perturbed by a sufficiently short wavelength which is stable at \( t = 0 \), according to LSA. In the CF case, this perturbation remains stable. In CV flow, however, the contact line stabilizes for early
times, just to become unstable at some later time when fluid thins sufficiently, so that this imposed wavelength becomes unstable. Similar “delay” in occurrence of instability was also observed experimentally [11,12,18-20], and discussed in a similar context in [7].

![Diagram](image)

**FIG. 7.** Length of the emerging patterns for different D’s, scaled by the initial length.

**IV. CONCLUSIONS**

In this work, we point out the main differences between the flow of a constant stream of a fluid (CF), and of a constant volume of a fluid (CV). While in both cases the characteristics of the emerging patterns are similar, there are also important differences. In particular, in the CV case, we never observe the formation of nonlinear traveling wave solutions, reported in [1]. Furthermore, thinning of the fluid can lead to instability; this effect is not present in the CF flow. We also show that the use of the CF profile as the inner solution of the CV case to describe the onset of the instability may not be appropriate if the instability develops for early times. In fact, the front structure (mainly the bump height) can be very badly approximated by CF profiles, thus leading to a wrong prediction of the dominant wavelength in the linear stage of the instability development.

The results presented here are the basis for further research in few different directions. First, we will analyze the influence which thinning of the fluid film has on the nonlinear mode interaction, as previously analyzed in the CF scenario [1]. Next, we plan to develop a complete LSA for the CV case in order to be able to predict the wavelength of the instability for wide domains such as those used in the experiments.

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