Self-similar solution of the second kind for a convergent viscous gravity current

Javier Alberto Diez and Roberto Gratton

Instituto de Física Arroyo Seco, Facultad de Ciencias Exactas, Universidad Nacional del Centro de la Provincia de Buenos Aires, Pinto 399, 7000 Tandil, Argentina

Julio Gratton

INFIP-CONICET, LFP, Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I, Ciudadd Universitaria, 1428 Buenos Aires, Argentina

(Received 11 June 1991; accepted 14 January 1992)

The axisymmetric flow of a very viscous fluid toward a central orifice is studied. In a recent paper, a self-similar solution for this problem has been found. The self-similarity is of the second kind and hence the flow remembers its initial condition only through a nondimensional constant which characterizes it. In this work this convergent flow is studied experimentally (using silicone oils) by measuring the front position and the height profile as a function of time. It is verified that the self-similar solution properly describes the flow within a certain interval of the cavity radius, where values are obtained for the similarity exponent $\delta$ in agreement (accounting for experimental errors) with the theoretical value 0.762... . The transition to the self-similar flow is also simulated numerically and numerical values are obtained for the time closure for different initial conditions. These simulations also show the theoretical self-similar flow after the cavity closure, which is very difficult to observe experimentally.

I. INTRODUCTION

Viscous gravity currents occur in many situations of interest for geophysics, industrial engineering, geology, and environmental sciences. The main feature of these currents is that the flow is primarily horizontal and is governed by a balance between gravity and viscous forces; inertia effects are negligible. In this context, when the length of the current greatly exceeds its thickness, the lubrication approximation holds and a nonlinear diffusion equation of the form

$$\frac{\partial h}{\partial t} = x^{-\alpha} \frac{\partial}{\partial x} \left( x^\alpha h^3 \frac{\partial h}{\partial x} \right)$$

(1)

describes the current profile $h(x,t)$. In Eq. (1), $x$ is the horizontal coordinate, $t$ is the time, and $\alpha=0,1,2$ stands for plane, cylindrical, and spherical symmetry, respectively. Equations of this kind are considerably important in mathematical physics because they serve to treat nonlinear diffusion and nonlinear heat conduction (radiative and electron transport in plasmas), as well as many other problems; for a more complete list see, for example, Seshadri and Na, Pert, Grundy, and Smyth.

The gravity current that we study in this paper is a converging flow of a very viscous liquid over a rigid horizontal surface. We produced this current by using a pool full of liquid that surrounds a small circular dam whose interior is dry. If the dam is suddenly removed, a gravity current will flow toward the center (the origin). The current has a converging front, which ultimately collapses at the origin.

This problem admits a self-similar solution of the second kind that has been studied theoretically in previous papers. In these works the similarity exponent $\delta$ and the thickness and velocity profiles have been found by solving a nonlinear eigenvalue problem and integrating numerically the self-similar equations. However, there are still some features of the solution that must be made clear. For instance, the solution contains a theoretically not determined coefficient, which takes values depending on the initial conditions; besides, it should be established where the self-similar solution is suitable for initial conditions of practical interest. The primary aim in this paper is to clarify these issues by carrying out experiments and numerical simulations of the problem.

This paper is organized as follows: in Sec. II we summarize the theory; in Sec. III we describe the experimental setup and techniques; and in Sec. IV we present a numerical treatment of the diffusion equation. Our main conclusions (Sec. V) are (a) the current effectively reaches a self-similar regime, as the front position versus time can be described by a power law in a relatively long interval of the radius; (b) this interval is approximately the last one-third of the initial radius of the dam; (c) the thickness profile in this region is in agreement with the theoretical results; and (d) the time closure (in terms of a convenient dimensional time) depends only on the geometrical shape of the initial thickness configuration.

II. THEORY

Let $H(x,t)$ be the thickness of the current, $\nu(x,z,t)$ the velocity, which we shall assume nearly horizontal, and $z$ the vertical coordinate; the no-slip boundary condition at the bottom of the current will be $\nu(x,z=0,t) = 0$, and at the
free surface we shall require \((\partial v/\partial x)_{x=0} = 0\). With these conditions and the above assumptions, a parabolic vertical velocity profile is obtained:

\[
v = 3ug(2H - z)/(2H^2)
\]  
(2)

in which \(v(x,t) = 2v(x,H) / 3\) is the average horizontal velocity. The only constant dimensional parameter that enters in the governing equations is \(g/\nu\) (\(g\) is the gravity and \(\nu\) is the kinematic viscosity coefficient), and it can be absorbed in the definition of a new dependent variable,

\[
h = (g/3\nu)^{1/3}H.
\]

Neglecting inertia and assuming that the pressure is nearly hydrostatic, one obtains the basic equations of the problem,

\[
\begin{align*}
\frac{\partial h}{\partial x} + v &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial (hv)}{\partial x} + \frac{ahv}{x} &= 0,
\end{align*}
\]

(4)

in which no constant dimensional parameter appears and \(\alpha = 1\). Here, it is assumed that the fluid is Newtonian and the surface effects have been neglected (the validity assumptions for the present experiments will be discussed later). Note that this system of equations can be reduced to the nonlinear diffusion equation (1). Equation (4) contains only quantities with dimensions of length, time, or combinations of both. Therefore \(h\) and \(v\) can generally be expressed as follows:

\[
h = (x^2 t^{-1/2} Z)^{1/3}, \quad v = x t^{-1} V.
\]

(5)

Here \(Z, V\) are dimensionless functions of \(x, t\) and of the constant parameters involved in the initial and/or boundary conditions. Self-similarity occurs if no more than one parameter \(b\) with independent dimensions appears in the problem. In general, \(b\) can be chosen so that its dimensions are \(b_L = LT^{-6}\), in which the similarity exponent \(\delta\) is a numerical constant. For self-similar motions, \(Z, V\) will be functions only of the similarity variable, \(\xi = x/ht^{6/5}\). Substitutions of Eq. (5) into Eq. (4) will yield two ordinary differential equations for the phase variables \(Z(\xi)\) and \(V(\xi)\), from which one can eliminate \(\xi\) to obtain an autonomous equation for \(V(Z)\):

\[
\frac{dV}{dZ} = \frac{(2\delta - 1)Z + 3(1 + \alpha)VZ + 3(\delta - 1)V}{3Z(2Z + 3V)}.
\]

(6)

Equation (6) and the remaining equation for \(\xi\), that is,

\[
\frac{d}{dZ} (\ln |\xi|) = - (2Z + 3V)^{-1},
\]

(7)

are the basic equations of the formalism. A self-similar problem is then essentially reduced to the integration (numerical in general) of Eq. (6), whose solutions are represented by integral curves in the \((Z, V)\) plane. For details of the study of the solutions of Eq. (6) in the phase plane, see Gratton and Minotti\(^9\) and Diez et al.\(^11\).

Here, we mainly deal with the velocity of the current near the closure time as well as near the front, i.e., in a very small region whose radius is small compared with the radius that characterized the initial configuration of the fluid. It is assumed that the cavity closure occurs at \(t = 0\). Therefore \(t < 0\) corresponds to times before the collapse, when the converging front is approaching the origin, and \(t > 0\) corresponds to times after the collapse, when the current has covered the initial dry region, and there is still flow tending to increase the thickness of the liquid in the central part.

In this situation we are left with no constant dimensionless governing parameters: those arising from the initial conditions do not give proper scales for the properties of the current in the region of interest, and the characteristic parameters of the flow are functions of time. Therefore the flow is self-similar, but the similarity exponent cannot be determined by simple dimensional analysis (because this analysis does not provide the value of the parameter \(b\)). We are in the presence of self-similarity of the second kind.

In previous papers\(^6,11\) the similarity exponent \(\delta\) for \(\alpha = 1, 2\) has been determined. Particularly, for \(\alpha = 1\) it is \(\delta = 0.762 651 1...\). Diez et al.\(^11\) also found the height and velocity profiles of the self-similar solution before and after the cavity closure.

III. EXPERIMENT

Experiments were carried out by using a circular Perspex basin of radius \(r_b = 20\) cm and a concentric circular dam of radius \(r_d = 5\) cm (see Fig. 1). We employed commercial silicone oil, whose viscosity was carefully measured (by using both Ostwold and Stokes viscometers) in the temperature range of \(18^\circ C < T < 24^\circ C\) and for velocity gradients ranging from \(5 \times 10^{-5}\) sec\(^{-1}\) to more than \(5\) sec\(^{-1}\), where deviations from Newtonian behavior were not observed. The oil temperature was recorded in each experiment to obtain the right value of the viscosity. The oil is placed in the annular region between the outer wall of the basin and the concentric circular retarding dam, which is connected to an opening device that quickly raises it up (aperture time \(\approx 0.1\) sec).

It is important to ensure accurately that the spreading surface is planar and horizontal. These controls were made by employing simple optical techniques. A good quality surface is desirable in order to avoid strong asymmetries in the flow which become evident by considerable deviation of the front shape from circularity. In fact, as the velocity is proportional to the gradient of \(H\), the flow is very sensitive to local deviations of the basin bottom from planarity. In our case, these deviations were somewhat less than \(0.1\) mm; this is good enough for fluid depths larger than \(2\) mm.

![FIG. 1. Scheme of the basin.](image-url)
In addition, we must also be sure that the surface forces do not play an important role in the experiments. From a dimensional point of view, there exists a characteristic thickness related to this force, namely $d = (\gamma / \rho g)^{1/2}$, $\gamma$ being the surface tension for the oil-air interface. As $\gamma \approx 20$ ergs cm$^{-2}$ for most of the oils, $d$ ($\approx 0.15$ cm) is of the order of the current thickness in our experiments; on this basis one could be led to think that surface tension might have a significant effect on the flow. However, it has been found experimentally\textsuperscript{12,13} that there exists a critical value of $\gamma$, namely $\gamma_c$, for every solid, such that any simple structural liquid which has a surface tension (with respect to air) $\gamma < \gamma_c$, completely wets the solid. It has been observed\textsuperscript{12} that $\gamma_c$ is essentially independent on the liquid, so that it may be considered as a parameter of the solid itself. It is known that $\gamma_c = 39$ ergs cm$^{-2}$ for Perspex, so we see that surface energy effects can be neglected as the oil completely wets the surface, i.e., the contact angle is zero. In fact, if this is the case, changes in the covered area do not contribute to changes in the surface energy. Thus simple geometrical considerations show that the ratio between the variations in the total surface energy and the total gravitational energy cannot exceed a quantity of the order $4(r_0 / r_f) (d / H)^2 (H / r_0)$ ($<1$ in our experiments), where $H$ is an average fluid height.

Because of the convergence of the flow, the parameters of the experiment (dam radius, basin radius, fluid depth, viscosity, etc.) must be chosen carefully in order that the flow reaches the adequate regime to observe the self-similar asymptotic solution. To begin with, we are interested in the flow within regions where $r < r_0$; at first sight this could be easily attained by taking a value of $r_0$ that is sufficiently large. However, to make them irrelevant in order to neglect the effects related to the presence of the external basin wall, it seems reasonable to require that the thickness $H_b = H(r_b)$ should remain nearly constant during the time of interest and consequently $r_b / r_0$ should be small. In the experiments, $r_b / r_0 = 0.25$ and to satisfy the condition $H_b \approx H_0$ within 1% accuracy, $r$ must be less than $0.1 r_0$, so that we may expect the similarity solution to hold for $r < 2$ cm.

Furthermore, the initial oil thickness $H_0$ in the annular region must be chosen properly for the flow to develop within the validity range of the lubrication approximation. Let us recall that its main hypotheses are (a) the length of the current is much greater than its depth, (b) the motion is essentially horizontal, and (c) the inertia effects are negligible.

Hypothesis (a) will remain valid if the radial length of the current is much greater than its mean thickness $H$. However, hypothesis (b) does not hold when the current is very close to the center ($r_f = 0$), where the vertical component $u_z$ of the velocity is not negligible. To satisfy hypothesis (c) we must consider the orders of magnitude of the gravity, inertial, and viscous forces in the flow, which are, respectively,

$$ F_g = \rho g H^2 R, \quad F_i = \rho U^2 H R, \quad F_v = \mu R^2 U / H, $$

$U$ being a mean speed and $R$ the radius of the current. Then, the Reynolds number is

$$ Re = F_i / F_v = (U H / \nu) (H / R). $$

In the lubrication approximation we must have $Re < 1$ and the gravity force is balanced by the viscous force, that is, $F_g = F_v$, whence $U = g H^2 / \nu R$. Consequently, the Reynolds number for a viscous current is

$$ Re = g H^4 / (\nu R)^2. $$

So, to maintain $Re < 1$ during the whole collapse, we must choose

$$ H_0 < (\nu^2 r_0 / g)^{1/5}. $$

In our case, for $\nu = 4.87$ cm$^2$ sec$^{-1}$ ($T = 20$ $^\circ$C), the initial depth must be much less than 1 cm.

The slow evolution of these flows ($u \leq 1$ cm sec$^{-1}$) allows the use of normal speed video recording (50 frames per second) as a suitable technique to observe the position of the front and the current profile. As the shape of the front presented small deviations from circularity (typically of the order of a few percent), it was necessary to develop an averaging procedure. The positions of the front on two orthogonal axes passing through the geometric center of the dam were measured at time intervals 0.1 sec $< \Delta t < 1$ sec, according to the case. For each time we calculated the centers and the radii of four circles, one for each of the three points. The averages of the centers and radii were taken as the center and radius of a circle approximating, though not exactly, the circular front of the current. The results obtained with this method are shown in Fig. 2, where the front radii versus time correspond to different initial oil depths, $H_0 = 2, 3, 4, 5$ mm.

As mentioned before, the self-similar solution can exist provided that the front radius $r_f$ is small enough so that the depth at the outer wall of the basin keeps almost constant, but not so small as to give $Re = g H^4 / (\nu r_f)^2 > 1$. Obviously, there is a certain degree of arbitrariness in the definition of the validity range, but a reasonable choice is

$$ 2H_0 < r_f < 0.1 r_0. $$

In Table I we give the values of $\delta$ obtained with this criterion.

![FIG. 2. Front position versus time obtained from video recording. The solid line of slope $\delta = 0.762$ is for comparison.](image)
TABLE I. Values of $\delta$ obtained with video recording (see Fig. 2).

<table>
<thead>
<tr>
<th>$H_0$ (cm)</th>
<th>$2H_0 &lt; r_f &lt; 0.1r_s$</th>
<th>Exp. points</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.710 ± 0.007</td>
<td>14</td>
</tr>
<tr>
<td>0.4</td>
<td>0.785 ± 0.005</td>
<td>31</td>
</tr>
<tr>
<td>0.3</td>
<td>0.734 ± 0.006</td>
<td>56</td>
</tr>
<tr>
<td>0.2</td>
<td>0.833 ± 0.006</td>
<td>61</td>
</tr>
</tbody>
</table>

where $\delta$ is the theoretical value of the similarity exponent, $\delta = 0.762...$.

We have also used an alternative technique to determine the average front radius, based on measuring by light collection the area $S(t)$ free of oil. In this manner, the average $r_f$ is simply taken as given by $r_f = [S(t)/\pi]^{1/2}$. This procedure improves the above described method, which averages only on four radii. However, the measurement becomes indirect and the current silicone oils cannot be used because of their high transparency. The technique was applied to a heavy oil, giving the results reported in Table II and Fig. 3. The corresponding value of $\delta$ was

$$\delta = 0.741 ± 0.066, \text{ that is, } \bar{\delta} = 0.972\delta.$$  (14)

If we disregard the case $H_0 = 0.2 \text{ cm}$, in which the shape of the front departed considerably from circularity, we obtain $\delta = 0.780 ± 0.003$, and then $\delta = 1.024\delta$.

To measure the current profile we recorded a side view of a straight luminous segment projected vertically on the current along a diameter. In Fig. 4 we compare the experimental points with the theoretical curve. As the theoretical values of $H$ are defined within an arbitrary constant factor, the curve was simply forced to pass through the farthest experimental point. It can be seen that the self-similar solution describes reasonably well the profile of the current.

IV. NUMERICAL SIMULATIONS

The converging flow we are considering is characterized by both a horizontal ($r_f$) and a vertical ($H_0$) scale. Thus it is convenient to introduce the nondimensional variables $X = r/ro$, $U = H/H_0$, $V = v/v_0$, $T = t/t_0$, where $v_0 = k H_0^2/x_0$ and $t_0 = x_0^2/kH_0^3$, with $k = g/3v$. Hence, Eq. (4) can be written as

$$\frac{\partial U}{\partial T} + X - \frac{a}{X^3} \frac{\partial}{\partial X} (X^a UV) = 0,$$  (15)

$$U \frac{\partial U}{\partial X} + V = 0,$$  (16)

and then Eq. (1) is

$$\frac{\partial U}{\partial T} = X - \frac{a}{X^3} \frac{\partial}{\partial X} \left(X^a U^3 \frac{\partial U}{\partial X}\right).$$  (17)

This equation was solved numerically in the domain $0 < X < X_b = r_f/ro$. Initially, the fluid will extend only over a portion of $X_b$, while the remaining part should be considered as "dry." However, as it turns out to be quite difficult to solve this moving boundary problem, we supposed that a very thin fluid layer covers the "dry" region, with a negligible thickness with respect to $H_0$ (typically, $10^{-3}H_0$).

TABLE II. Values of $\delta$ obtained by measuring the area $S(t)$ free of oil in the central region of the basin (see Fig. 3).

<table>
<thead>
<tr>
<th>$H_0$ (cm)</th>
<th>$2H_0 &lt; r_f &lt; 0.1r_s$</th>
<th>Exp. points</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.781 ± 0.003</td>
<td>13</td>
</tr>
<tr>
<td>0.4</td>
<td>0.776 ± 0.004</td>
<td>15</td>
</tr>
<tr>
<td>0.3</td>
<td>0.783 ± 0.006</td>
<td>18</td>
</tr>
<tr>
<td>0.2</td>
<td>0.626 ± 0.006</td>
<td>13</td>
</tr>
</tbody>
</table>

![FIG. 3. Front position versus time obtained by measuring the area $S(t)$ free of oil in the central region of the basin. The solid line of slope $\delta = 0.762$ is for comparison.](image)

![FIG. 4. Theoretical and experimental thickness profiles before the cavity closure.](image)
presence of this thin layer does not invalidate the numerical results as was tested by computing the one-directional spreading of a constant mass of fluid, which has a well-known self-similar solution of the first kind (for instance, see Pattle). Therefore, if \((0,X_b)\) is the integration interval, Eq. (17) must be integrated with the initial conditions: \(U = U_0(X)\) in the region \((1,X_b)\), and \(U=0\) in the rest, i.e., \((0,1)\). The boundary conditions must be

\[
\frac{\partial U}{\partial X} \bigg|_{X=0} = \frac{\partial U}{\partial X} \bigg|_{X=X_b} = 0 \tag{18}
\]

in such a way that the mass flow is zero at the boundaries.

In order to avoid nonlinearities involving the derivatives (which are harder to handle than nonlinearities in the function itself), it is better to transform Eq. (17) by substituting \(\omega = U^4\). Thus one obtains a more convenient expression:

\[
\frac{\partial \omega}{\partial T} = \omega^{3/4} \frac{\partial \omega^2}{\partial X^2} + \frac{\alpha \omega^{3/4}}{X} \frac{\partial \omega}{\partial X}. \tag{19}
\]

We integrated Eq. (19) by applying an implicit scheme of weighted time, where the spatial derivatives are expressed as a weighted average of \(\omega\) between time step \(n\) and \(n+1\). The nonlinearity in \(\omega\) is expressed as a linear extrapolation defined also as a weighted average of \(\omega\) between time step \(n-1\) and \(n\), thus generating a three-level scheme. Therefore, the scheme which approximates Eq. (19) for a generic node of the grid, is

\[
\frac{\omega_i^{n+1} - \omega_i^n}{\Delta T} = \frac{(\omega_i^{n+1/2})^{3/4}}{\Delta X^2} \left[ \vartheta (\omega_{i+1}^{n+1} - 2\omega_i^{n+1} + \omega_{i-1}^{n+1}) \right] + (1 - \vartheta) (\omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n)
+ \frac{\alpha}{X_i} \frac{(\omega_i^{n+1/2})^{3/4}}{2\Delta X} \left[ \vartheta (\omega_{i+1}^{n+1} - \omega_i^{n+1}) \right] + (1 - \vartheta) (\omega_{i+1}^n - \omega_{i-1}^n), \tag{20}
\]

where

\[
\omega_i^{n+1/2} = \vartheta \omega_i^n - \frac{1}{\vartheta} \omega_i^n - 1, \tag{21}
\]

\(\vartheta\) is a weighing coefficient between 0 and 1 and \(\Delta X = L/(N-1), N\) being the number of points in the grid including the extremes. Different values of \(\vartheta\) determine a family of numerical schemes; for instance, for \(\vartheta = 0, 1, \) and \(\frac{1}{2}\) we obtain the explicit, implicit, and Crank–Nicolson schemes. This general scheme is consistent with the differential equation [Eq. (19)]. The order of consistency is \(O(\Delta X^2, \Delta T)\).

Note that with this scheme the velocity \(v\) does not play any role in the numerical calculations, but is obtained afterwards by using the relation,

\[
V_i^t = -(U_i^t)^2 (U_{i+1}^t - U_{i-1}^t)/2\Delta X, \tag{22}
\]

where \(U_i^t = (\omega_i^t)^{1/4}\).

In comparing the numerical results with the theoretical solution [Eq. (5); see Diez et al.], it must be kept in mind that the self-similar solution corresponds to the case \(X_s = \infty\). However, numerical calculations with different values of \(X_b\) have shown that the computed solution does not depend on \(X_b\), provided that \(X_b > 3 - 4\). In the following we will use the time variable \(\tau = t - t_*\), where \(t_*\) is the cavity closure time and we shall take \(\alpha = 1\).

A. Results before the closure of the cavity (\(\tau < 0\))

One of the clearest ways to observe the flow transition to the self-similar regime is to study the front position as a function of time. In a dimensional form, according to the self-similar theory, this is given by

\[
r_f = \xi_f b (-\tau)^{\delta} \quad (\tau < 0), \tag{23}
\]

where \(\xi_f\) is a dimensionless constant. The dimensional constant \(b\) can be chosen as

\[
b = r_0/4t_0 = r_0^{-28}(\chi H \delta / 3 \nu)^4, \tag{24}
\]

but \(\xi_f\) cannot be obtained from the theory. Its value, which depends on the particular choice of initial conditions of the problem, can only be obtained from the asymptotics of the complete numerical simulation. This is so because the asymptotic flow does not completely "forget" the initial conditions, but it selects from them a number \(\xi_f\) which characterizes these conditions (see, for instance, Barenblatt).

The comparison of the front position obtained numerically with Eq. (23), allows both to verify the similarity exponent \(\delta\) and to obtain the value of \(\xi_f\) for each particular problem. In Fig. 5 it is shown \(X_f\) as a function of \((1 - t/t_*) = (-\tau/t_*)\). If we approximate linearly this curve in the range \(0.1 < X < 0.35\), we obtain

\[
X_f = (0.702 \pm 0.001)(-\tau/t_*)^{0.76 \pm 0.003}, \tag{25}
\]

which corresponds to a value of \(\delta\) less than the theoretical value by only 0.2%. In this way, as we observed in the experiments, we see that the front motion is well described by the self-similar solution for radii less than one-third of the dam's radius. The numerical simulation points reported in Fig. 5, which correspond to very small \(X\) values, do not fall on the straight line because the error in the determination of the

\[FIG. 5. Front position versus time from the numerical simulations.\]
TABLE III. Closure times \( \Gamma \) obtained for different initial conditions. Variable \( \varphi \) is defined by: \( \varphi = \{ 1 - [(X_b - X)/(X_b - X_c)] \}^{1/3} \).

<table>
<thead>
<tr>
<th>Initial condition</th>
<th>( \varphi )</th>
<th>( \Gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( u = 2 - \varphi )</td>
<td>0.8772</td>
<td></td>
</tr>
<tr>
<td>(b) ( u = 1.4 - 0.4\varphi )</td>
<td>1.336</td>
<td></td>
</tr>
<tr>
<td>(c) ( u = 1.845 )</td>
<td>1.845</td>
<td></td>
</tr>
<tr>
<td>(d) ( u = \sqrt[3]{30\varphi} )</td>
<td>2.222</td>
<td></td>
</tr>
<tr>
<td>(e) ( u = \varphi )</td>
<td>3.874</td>
<td></td>
</tr>
</tbody>
</table>

front position becomes important for these small radii (of the order \( \Delta X = 0.01 \)).

The value of \( \xi_f \) can be obtained from Eqs. (23)-(25), considering that for \( t = 0 \), that is, \( \tau = \tau_c \), we obtain

\[ \xi_f \Gamma = 0.702, \quad (26) \]

where \( \Gamma \) is the closure time in units of \( t_0 \) as given by the numerical simulations, that is, \( \Gamma = t_c/t_0 \). In particular, for the steplike initial condition [\( U = 1 \), for \( X > 1 \)], case (c) of Table III], we have \( t_c = 1.8452 \), so that \( \xi_f = 0.440 \). This time was taken at the moment in which the thickness at \( X = 0 \) reaches \( 0.1H_0 \), because the numerical simulation does not produce exactly vertical fronts.

As said before, \( \xi_f \) (or equivalently, \( t_c \)) depends on the choice of the initial conditions, which is easy to set up in the numerical calculations, but it is difficult to control in the experiments. For instance, the experimental closure times for \( H_0 = 0.2, 0.3, 0.4 \), and 0.5 cm, are \( \Gamma = t_c/t_0 = 2.42, 2.70, 2.31 \), and 2.15. We think that the difference in closure time between numerical calculations and experiments is mainly due to the fact that the initial profile that is set up after removing the dam is not a step profile, but it deviates from it over a certain distance by less than order \( H_0 \). This effect may be important even if \( H_0 < t_0 \), because of the sensitivity of the self-similar solutions of the second kind with respect to details in the initial conditions. In Table III we show this effect by comparing the nondimensional closure times \( \Gamma = t_c/t_0 \) computed for different initial height profiles. Besides, Fig. 6 shows that the experimental points (from Fig. 2) \( r_f/r_0 \) versus \( (1 - t/t_c) \) are poorly fitted by the numerical curve obtained starting from a steplike profile; in contrast, the fit is good for a smooth profile such as case (e) of Table III.

In that table we can also see that those profiles whose thickness is always below the steplike function give larger values of \( t_c \) than those which are above. This is because when the profile is below the steplike one, the fluid has on the average a longer way to go (greater total dissipation) and it also has less potential energy to drive it.

In conclusion, the closure time is determined only by the geometry of the initial condition, which also selects the asymptotics to which the flow converges, that is, the value of \( \xi_f \).

In order to compare the numerical profiles of thickness and velocity with the self-similar solution, we write the latter as follows:

\[ U(\tau/X_f^2)^1/3 = [\eta^2 Z(\eta)]^{1/3}, \quad V\tau/X_f = \eta V(\eta). \quad (27) \]

The functions \( U(\tau/X_f^2)^1/3 \) and \( V\tau/X_f \) vs \( \eta \) (as given by numerical simulations) tend asymptotically to the right-hand sides of Eq. (27). This can be seen in Figs. 7 and 8 for several values of \( \tau < 0 \), according to the asymptotic character of the self-similar solution.

B. Results after the closure of the cavity (\( \tau > 0 \))

For times larger than \( t_c (\tau > 0) \), but close to it (\( \tau = 0) \), the flow is described by another self-similar solution, which
has the same similarity exponent \( \delta \) as the solution for \( \tau < 0 \)
(see Diez et al.\textsuperscript{11}). Unlike that solution, the flow has a self-
similar behavior at the beginning of this stage (\( \tau = 0 \)), and
after some time it should depart from the self-similarity.

A remarkable characteristic of the self-similar solution
for \( \tau > 0 \) is the existence of a maximum in the velocity profile,
or better said, in the function \( \eta V(\eta) \) vs \( \eta = r/\xi \beta \tau^{\delta} \) (see Fig.
5 of Diez et al.\textsuperscript{11}). This means that there is a circle of maxi-
mum velocity at \( r = r_M \) which expands outwards according to

\[
r_M = \eta_M \xi \beta \tau^{\delta} \quad (\tau > 0),
\]

where \( \delta = 0.762... \) and \( \eta_M = 2.656 15 \) is the correspond-
ing value of \( \eta \). In Fig. 9 we represent \( X_M = r_M/r_0 \) versus
time as given by the numerical simulations, as well as by Eq.
(28). It can be observed that the self-similarity holds for
\( r < r_0/3 \), as it was observed for the convergent front motion.

If we redefine the self-similar variable \( \eta \) in the form
\( \lambda = r/r_M = \eta/\eta_M \), the comparison between the numerical
and theoretical profiles of thickness and velocity can be done
as before by plotting \( U(\tau/X_M^{1/3})^{1/3} \) and \( V_\tau/X_M \) vs \( \lambda \). In Figs.
10 and 11, we show the profiles for \( T = 1.9 \ (\tau = 0.055) \).
Note that even for these small values of \( \tau \), the deviation be-
tween the numerical solution with respect to the self-similar
one, becomes evident.

We have also compared the evolution of the thickness at
the center of symmetry with the self-similar law. According
the numerical calculations \( U_0 = U(0,T) = 0 \) for \( \tau = 0 \)
and \( U_0 \to 1 \) for \( \tau \to \infty \). On the other hand, the self-similar
solution gives (for \( \tau \) close to zero)

\[
U_0 = (\xi/\gamma N)^{1/3} \tau^\gamma,
\]

where \( \gamma = (2\delta - 1)/3 = 0.174 74... \) and \( N \) is a matching
constant between the solution before and after the cavity
 closure. In Fig. 12 we show the time evolution of \( U_0(T) \)
FIG. 12. Numerical (○) and theoretical [Eq. (29)] thickness at the origin versus time.

obtained numerically and the straight line given by Eq. (29). There we can see that the power-law behavior is valid even up to relatively large thickness ($\approx 0.7H_0$).

V. SUMMARY AND CONCLUSIONS

We have shown both experimental and numerically the existence of the self-similar solution of the second kind for the converging viscous flow. As a result, we can say that the self-similar solution (that is, the right power-law dependence of the front motion on time and a good agreement with the theoretical thickness profile) can be observed in a limited region of the space, i.e., $r < 0.4r_0$. This fact is not surprising since the theoretical solution was obtained for an infinite pool and for front radii much smaller than the dam radius. However, the region of validity of the solution cannot be determined by the theory and only the experiments or the numerical simulation can show it. Here, we have given a reasonable criterion to determine this region which gives a validity range much larger than may be expected from the mathematical approximations. In particular, the numerical simulations have shown that the time evolution of the thickness at the center can be described by the self-similar solution even for values of $H$ of the order of $H_0$.

The numerical simulations have been particularly useful to describe the flow after the closure. As a result, we have shown that the time closure (in terms of a convenient dimensional time) depends only on the geometrical shape of the initial thickness configuration. This dependence cannot be established experimentally as the initial conditions for the viscous stage of the flow is not controllable and is not known a priori.

ACKNOWLEDGMENTS

This work was supported in part by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina.
